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REGULAR SEPARABLE GRAPHS OF MINIMUM ORDER WITH GIVEN DIAMETER

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A (d, c, v)-graph G is one which is regular of degree v and has diameter d and connectivity c. G is said to be minimum if it is of minimum order, i.e. has the minimum number of points; G is separable if $c = 1$.

In this paper, the minimum order of a $(d, 1, v)$-graph is determined and the construction of all minimum $(d, 1, v)$-graphs is described.

1. Introduction

The *order* of a graph G is the number of its points, and the *degree* (or *valence*) *of a point* p_i in G is the number of lines incident with p_i. If all of its points have the same degree v, G is said to be *regular* of degree v. G has *connectivity* c, or is *c-connected*, if c is the minimum number of points whose removal from G results in either a disconnected graph or the trivial graph (a graph of order one, which therefore has no lines). A shortest path joining two points p_i, p_j is called a p_i–p_j *geodesic*, and the *diameter* of G is the length of any longest geodesic in G.

A minimum (d, c, v)-graph G is one which is regular of degree v, has diameter d and connectivity c, and is of minimum order, i.e. has the minimum number of points. Such graphs constitute a sub-class of those addressed in an extensive literature on the interrelationships between connectivity, valencies and diameter of a graph [1–45].

As Klee and Quaife [28] have noted, (d, c, v)-graphs are of practical as well as mathematical interest since removing fewer than c of the points (and hence certainly fewer than c of the lines) from such a graph G does not disconnect it. G thus describes a survivable communications or transportation network.

Klee and Quaife have classified and enumerated all minimum $(d, 1, 3)$-graphs and all minimum $(d, 2, 3)$-graphs, using an ad hoc approach to the problem [27]. Myers [40] subsequently re-articulated their minimum $(d, 1, 3)$-graph results through a mathematical machine (lemmas and theorems) formulation, and has also shown how to classify and construct all minimum $(d, 3, 3)$-graphs, though he was not able to enumerate them [41]. Klee has classified and enumerated the minimum $(d, 3, 3)$-graphs of odd diameter [29].

After Klee and Quaife [28], a c-connected graph of diameter $\geqslant d$ in which each point has degree $\geqslant v$ is called a $\langle d, c, v\rangle$-graph. They have determined the order of a minimum $\langle d, c, v\rangle$-graph and have given a construction of the same. They have apparently also determined the minimum orders of (d, c, v)-graphs [30], and they note that the minimum orders of $\langle d, c, c\rangle$-, $(d, 3, 3)$-, and $\langle d, 1, v\rangle$-graphs have been determined previously by Grünbaum and Motzkin [21], Klee [26], and Moon [35].

In the present paper, we determine the orders of the blocks in a minimum $(d, 1, v)$-graph and we show how to classify and construct some such graphs for all values of v and d. Enumeration of them is not, however, considered.

Except where noted, our terminology follows Harary [24].

2. The blocks of a $(d, 1, v)$-graph

A $(d, 1, v)$-graph G has connectivity $c = 1$ and hence has at least one cutpoint. The cutpoints of G separate the graph into maximally connected *components*. A component of order 2 has exactly one line, which is called a *bridge* of G. The components of G which are of order >2 are called *blocks*. A block which has exactly one point p_0 of degree less than v but not less than 2, p_0 thus a cutpoint of G, is called an *end-block*, denoted by B_x where x is the degree of the point p_0, so that $2 \leq x \leq v-1$. A block which is not an end-block is called a *cut-block*, denoted by $A_{x_1,x_2,\ldots,x_n}$ where n is the number of cutpoints of G that are in the cut-block, and where $x_1, x_2, \ldots, x_n$ are the degrees of those cutpoints in $A_{x_1,x_2,\ldots,x_n}$. For example, the minimum $(10, 1, 3)$-graph in Fig. 1 has two end-blocks B_2, a cut-block $A_{2,2}$, and two bridges. We call a $(d, 1, v)$-graph G a *string* if each of its cutpoints separates G into exactly two components (each of which may also be separable). The graph of Fig. 1 is thus a string.

The *diameter of an end-block* B_x is the maximum of the distances from its point of degree x to all other points in B_x. The distance apart of any two points p_i, p_j each of degree less than v in a cut-block $A_{x_1,x_2,\ldots,x_n}$ is called the p_i–p_j *diameter of the cut-block* (or simply, the diameter when $n = 2$), there thus being $\frac{1}{2}n(n-1)$ diameters in the cut-block. Since every graph has an even number (which may be zero) of points of odd degree, and since an end-block B_x of order n of a $(d, 1, v)$-graph has $(n-1)$ points of degree v and 1 point of degree $x < v$, we have:

Lemma 1. *There is no end-block* B_{2k+1} $(1 \leq k \leq \frac{1}{2}(v-2))$ *in a* $(d, 1, v)$*-graph of even degree* v.

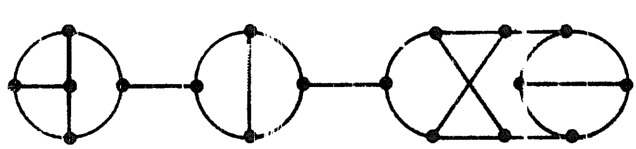


Fig. 1. A minimum $(10, 1, 3)$-graph.

In particular:

Theorem 1. *Every graph and therefore every $(d, 1, v)$-graph of even degree v is bridgeless.*

It is immediate from the definitions of an end-block and its diameter that:

Lemma 2. *There is no end-block with diameter* 1.

Minimum end-blocks of diameter 2

Lemma 3. *Every minimum end-block B_x $(2 \leq x \leq v-1)$ of diameter* 2 *is of order $v+2$ for x even, and is of order $v+3$ for x odd.*

Proof. Let p_0 be the point of degree x in B_x and let p_2 be a point a distance 2 from p_0. Since p_2 is of degree v and since the number of its points of odd degree must be even, B_x is of order at least $v+2$ when x is even and at least $v+3$ when x is odd (v then also being odd). We show by construction that a B_x of minimum order $v+2$, $v+3$ exists when x is even, odd, respectively, as follows. For the case when x is even we first construct the complete graph K_v of order v. Since $2 \leq x \leq v-1$ and x is even there is at least one set $E(x/2)$ of $\frac{1}{2}x$ point-disjoint lines in K_v. Let $B' = K_v - E(\frac{1}{2}x)$, so that B' is of order v, has x points of degree $v-2$, and has $v-x$ points of degree $v-1$. The graph obtained on adjoining a point p_0 to each of the x points of degree $v-2$ and a point p_2 to all of the points of B' is a minimum end-block B_x. For the case when x is odd, so that v is also odd, we first let $J_x = K_x - E(C_x)$, the regular graph of odd order x and even degree $x-3$ obtained on removing the x edges $E(C_x)$ of the cycle C_x from the complete graph K_x of order x. And let H_{v-x} be the regular graph of even order $v-x$ and even degree $v-x-2$ obtained on removing $\frac{1}{2}(v-x)$ point-disjoint lines from K_{v-x}. Let $B'' = J_x + H_{v-x}$, so that B'' is of order v. The graph obtained on adjoining a point p_0 to each of the x points of J_x in B'' and adjoining each of two points p_2, p_2' to all v points of B'' is a minimum end-block B_x of order $v+3$ with diameter 2.

Figs. 2, 3 show the construction of B_4, B_3, respectively, for $v = 5$.

Minimum end-blocks of diameter 3

Lemma 4. *Every minimum end-block B_x $(2 \leq x \leq v-1)$ of diameter* 3 *is of order $v+x+2$.*

Proof. Let p_0 be the point of degree x in a minimum end-block B_x of diameter 3 and let p_3 be a point a distance 3 from p_0, so that p_3 is of degree v. Then $B_x - p_0 - p_3$ is of degree at least $v+x$, so that B_x is of degree at least $v+x+2$. We show by construction that B_x is of degree exactly $v+x+2$. First consider the case when v is even, so that x and $v+x$ are also even. It is well known [24] that the

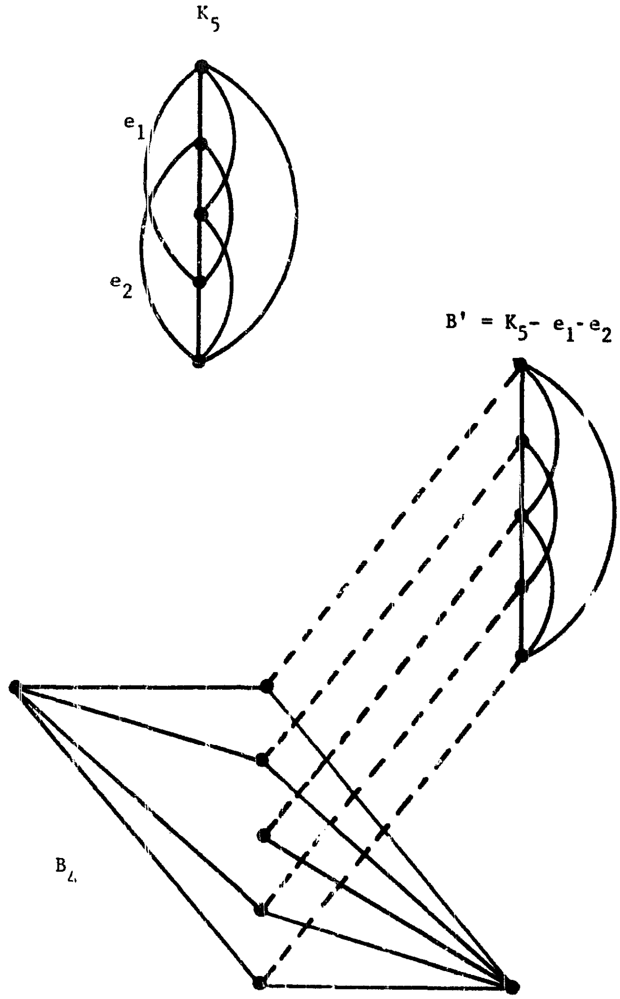


Fig. 2. Construction of minimum end-block B_4 with diameter 2 for $v = 5$.

complete graph K_{2n} of even order $2n$ is the line-disjoint union of n factors, $n-1$ of which are of degree 2, the other of degree 1. For example, the three factors of K_6 are as shown in Fig. 4(a). Let B'_{v+x}, where $v=2y$ and $x=2z$, be the regular graph of odd degree $(v+x)-3$ and order $v+x$ which is the subgraph $B'_{v+x} = K_{v+x} - F_2$ of K_{v+x} where F_2 is one of the factors of degree 2 of K_{v+x}. The factor F_1 of degree 1 of K_{v+x}, and therefore also of B'_{v+x}, is a set of $\frac{1}{2}(v+x) = y+z$ point-disjoint lines. Form the graph B_x by adjoining a point p_0 to x of the points of B'_{v+x} and adjoining a point p_3 to the other v points of B'_{v+x} so that B_x is of order $v+x+2$. B_x is seen to be a minimum end-block of diameter 3. For example, the minimum end-block B_2 for $v=6$ is as shown in Fig. 5(a). When V is odd and x is also odd, so that $v+x$ is even, B_x is constructed from B'_{v+x} exactly as above. When v is odd and x is even, so that $v+x$ is odd, we first form a regular graph B''_{v+x} of even degree $v-1$ as follows. The complete graph K_{v+x} of odd

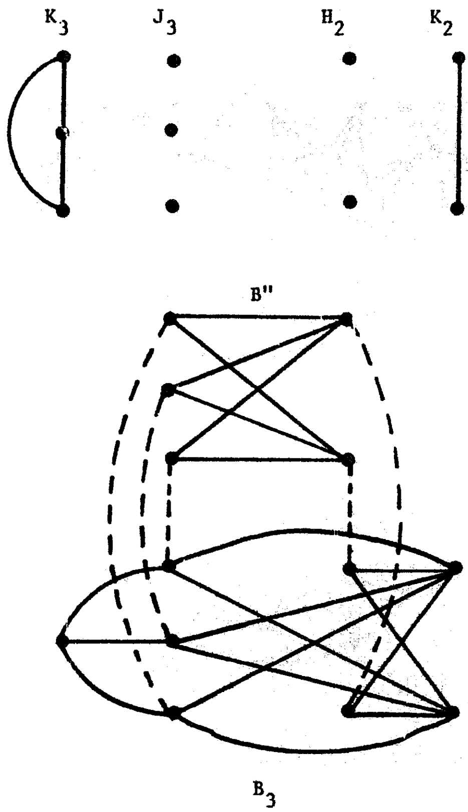
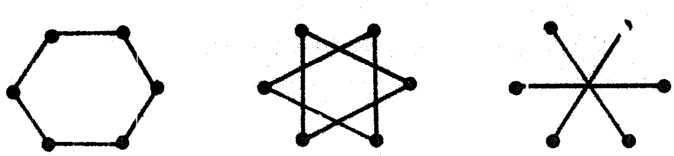
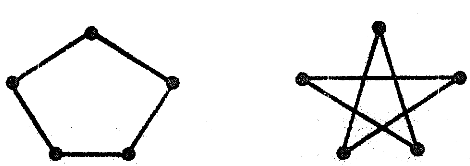


Fig. 3. Construction of minimum end-block B_3 of diameter 2 for $v = 5$.

degree $v+x=2n+1$ has n spanning factors each of degree 2. For example, the two factors of x_5 are shown in Fig. 4(b). Let B''_{v+x} be the spanning subgraph of K_{v+x} which is the point-disjoint union of $\frac{1}{2}(v-1)$ of these factors. Then B_x is formed by adjoining a point p_0 to x of the points of B''_{v+x} and a point p_3 to the other v of its points, illustrated by B_2 in Fig. 5(b) for $v=5$. The lemma follows.

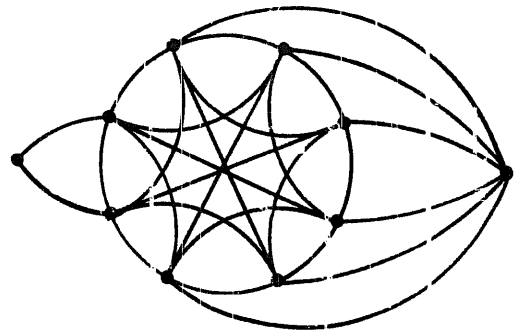


(a) The factors of K_6

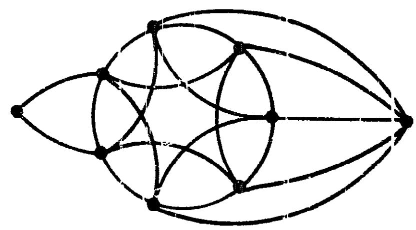


(b) The factors of K_5

Fig. 4. Factorization of complete graphs.



(a) B_2 for $v = 6$



(b) B_2 for $v = 5$

Fig. 5. Construction of minimum end-blocks of diameter 3.

Minimum end-blocks of diameter 4

Lemma 5. *Every minimum end-block B_x of diameter 4 is of order $2v+2$ for all v when x (and therefore also v) is odd, and is of order $2v+2$ for v even and $2v+3$ for v odd when x is even.*

Proof. Let p_0 be the point of degree x in B_x. Denote by P_i $(0 \leq i \leq 4)$ the set of points that are a distance i from p_0 so that $|P_0| = 1$, $|P_1| = x$, $|P_3| \geq v$, and $|P_4| \geq 1$. Since each point in P_i $(i > 0)$ is of degree v, P_2 is of order $|P_2| \geq v - x$ $(2 \leq x \leq v-1)$. The order of B_x is the thus at least $2v+2$ when x (and therefore v) is odd or when both x and v are even, and $2v+3$ when x is even and v is odd (so that there are then an even number $2v+2$ of points of odd degree v). We show by construction that a B_x of the minimum order $2v+2$ or $2v+3$, for the respective cases, exists. First consider the case when x, and therefore v, is odd. Let B' be the graph obtained on adjoining a point p_0 to each of the x points of K_x in the join $K_x + K_{v-x}$ of the complete graphs K_x, K_{v-x}. Denote by P_1, P_2' the set of x, $v-x$ points of K_x, K_{v-x}, respectively, in B', so that $|P_1| = x$, $|P_2'| = v - x$, and each point in P_1, P_2' is a distance 1, 2, respectively, from p_0. Each point of P_2' is of degree $v-1$ and each point of P_1 is of degree v in B', and B' is of order $v+1$. Now let B''' be the minimum end-block B_{v-x} of diameter 2 which has the point p_0''' of even degree $v-x$, its other points being of degree v. By Lemma 3, B_{v-x} is of order $v+2$. Let $B'' = B''' - p_0'''$, so that B'' has $v-x$ points of degree $v-1$. The graph obtained on adjoining the $v-x$ points of B' and the $v-x$ points of degree $v-1$ in B'', in pairs, is an end-block B_x of minimum degree $2v+2$. The case when v and x are both even is proved exactly as above, the minimum B_x again being of order $2v+2$. And when v is odd and x is even, the only difference in the proof is that $|P_2|$ is then $v-x+1$, an increase of 1 over the previous two cases, so that the technique again applies and yields $2v+3$ for the minimum-order B_x.

The minimum end-block B_3 of diameter 4 for $v=7$ as constructed by this technique is shown in Fig. 6.

Minimum end-blocks of diameter 5.

Lemma 6. *Every minimum end-block B_x of diameter* 5 *has a cut-point and is of order* $2v+3$ *when x is even and* $2v+4$ *when x is odd* $(2\leqslant v-1)$.

Proof. Let p_0 be the point of degree x in B_x and denote by P_i $(0\leqslant i\leqslant 5)$ the set of points a distance i from p_0. As in the proof of Lemma 5, $|P_0|=1$, $|P_1|=x$, $|P_2|\geqslant v-x$, $|P_3|>0$, $|P_4|\geqslant v$, and $|P_5|\geqslant 1$, from which it is deduced that B_x is of order at least $2v+3$ when x is even and $2v+4$ when x is odd. As before, we establish the lemma by constructing a B_x of the minimum order $2v+3$ or $2v+4$, as the case may be. We first construct the graph B' as in the proof of Lemma 5, and then adjoin a point p_3' to each of the $v-x$ points of the complete subgraph

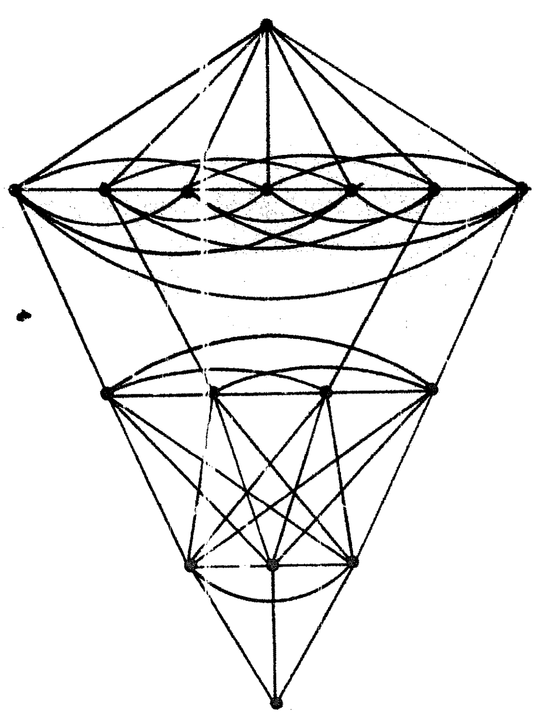


Fig. 6. Minimum end-block B_3 of diameter 4 for $v=7$.

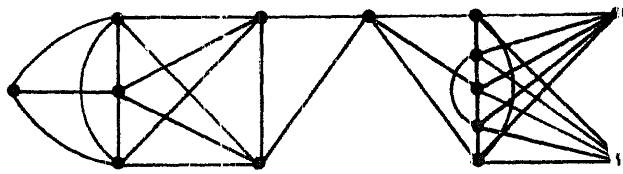


Fig. 7. Minimum end-block B_3 of diameter 5 for $v=5$.

K_{v-x} of B' so that p_3' has degree $v-x$ in B'. Let B'' be the minimum end-block B_x of diameter 2 in which p_3'' is the point of degree x, all of the other points in B'' being of degree v. We form the end-block B_x of diameter 5 by simply coalescing the point p_3' of B' with the point p_3'' of B'' into a single point p, which is then a cut-point of B_x. Since $B'-p_3'$ is of minimum order $v+1$ and B'' is of minimum order $v+2$ when x is odd and $v+3$ when x is even, B_x is of minimum order $2v+3$ when x is odd and $2v+4$ when x is even. The lemma follows.

The minimum end-block B_3 of diameter 5 and order $2v+4=14$ for $v=5$ constructed by this technique is shown in Fig. 7.

In Fig. 7, the subgraph B' is of order 7, has one point of degree 3, one of degree 2, and five of degree $v=5$, and is a *cut-block* $A_{3,2}$ of diameter 3 for $v=5$ which is of minimum order. A minimum end-block B_x of diameter $b>4$ can similarly be constructed as a string of cut-blocks, and an end-block of diameter less than b. It is thus not necessary to consider end-blocks of diameter $b>5$. The results of Lemmas 3–6 summarize as follows:

Theorem 1. *There is no end-block B_x of diameter* 1. The order of a minimum end-block B_x $(1\leqslant x\leqslant v-1)$ *of diameter b* $(2\leqslant b\leqslant 5)$ *as a function of v is as given in Table* 1.

A (not necessarily minimum) $(d, 1, v)$-graph with $d\leqslant 4$ results on either stringing an end-block B_κ $(2\leqslant k\leqslant v-2)$ of diameter b_1 $(\geqslant 2)$ with an end-block B_{v-k} of diameter $b_2=d-b_1$, or (for $d>4$) stringing a bridge and two end-blocks B_{v-1}, one of diameter b_1 $(\geqslant 2)$ and the other of diameter $b_2=d-b_1-1$. We list the sum

Table 1. Order of minimum end-block B_x of diameter b

	v even	c odd	
b	x even	x even	x odd
2	$v+2$	$v+2$	$v+3$
3	$v+x+2$	$v+x+2$	$v+x+2$
4	$2v+3$	$2v+3$	$2v+2$
5	$2v+3$	$2v+3$	$2v+4$

Table 2. Sum of orders of two end-blocks B_{x_1}, B_{x_2} ($x_1+x_2=v$ and $x_1=x_2=v-1$) of respective diameters $b_1, b-b_2$

		v even	v odd		
		$x_1+x_2=v$	$x_1+x_2=v$		
b	b_1, b_2	x_1 even x_2 even	x_1 even x_2 odd	x_1 odd x_2 even	$x_1=x_2$ $=v-1$
4	2, 2	$2v+4$	$2v+5$	$2v+5$	$2v+4$
5	2, 3	$2v+x_2+4$	$2v+x_2+4$	$2v+x_2+5$	$3v+3$
6	2, 4	$3v+5$	$3v+4$	$3v+6$	$3v+5$
6	3, 3	$3v+4$	$3v+4$	$3v+4$	$4v+2$
7	2, 5	$3v+5$	$3v+6$	$3v+6$	$3v+5$
7	3, 4	$3v+x_1+5$	$3v+x_1+4$	$3v+x_1+5$	$4v+4$
8	3, 5	$3v+x_1+5$	$3v+x_1+6$	$3v+x_1+5$	$4v+4$
8	4, 4	$4v+6$	$4v+5$	$4v+5$	$4v+6$
9	4, 5	$4v+6$	$4v+7$	$4v+5$	$4v+6$
10	5, 5	$4v+6$	$4v+7$	$4v+7$	$4v+6$

of the orders of two minimum end-blocks B_{x_1}, B_{x_2} ($x_1+x_2=v$ and $x_1=x_2=v-1$) of diameters b_1, $b_2=b-b_1$, respectively, in Table 2 as a function of v, x_1 and x_2. We observe that in those cases when it is a function of x_1 or of x_2, the sum is minimum when $x_1=2$, $x_2=2$ when x_1, x_2 are even and $x_1=3, x_2=3$ when they are odd.

We shall refer to Table 2 later on in discussing $(d, 1, v)$-graphs of minimum order. For example, since a $(4, 1, v)$-graph G cannot have a bridge, G is minimum when it is a string of two minimum end-blocks each of diameter 2. From Table 2, the order of a minimum $(4, 1, 4)$-graph ($v \geq 4$) is thus $2v+3$ for v even and $2v+4$ for v odd.

3. Minimum cut-blocks

We shall show later on that every minimum $(d, 1, v)$-graph G is a string, so that each cut-block of G is a cut-block A_{x_1,x_2} ($2 \leq x_1 \leq v-1$; $2 \leq x_2 \leq v-1$); i.e., if A_{x_1,x_2} is of order n, it then has one point p_0 of degree x_1, one point p_a of degree x_2, and $n-2$ points of degree v, the points p_0 and p_a being cut-points of G. The diameter a of A_{x_1,x_2} is the distance between P_0 and p_a in A_{x_1,x_2} and in G.

Denote by p_i the set of points that are a distance i (≥ 0) from p_0 in A_{x_1,x_2}, so that $|P_0|=1$, $|P_1|=x_1$, and $|P_a| \geq 1$. It is observed that $i>a$ is possible when $a=1$, illustrated by the minimum cut-block $A_{2,2}$ for $v=3$ in Fig. 8 which has $a=1$ and $0 \leq i \leq 2$, with $|P_0|=1$, $|P_1|=2=|P_a|$, and $|P_2|=3$. It will become apparent later, however, that a cut-block of diameter 1 is of no interest since a minimum $(d, 1, v)$-graph G has no such cut-block.

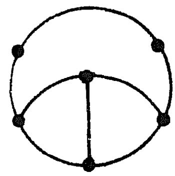


Fig. 8. Minimum cut-block $A_{2,2}$ of diameter 1 for $v = 3$.

Natural cut-blocks

We construct a special class of minimum cut-blocks $A_{x,y}$, called *natural cut-blocks*, of which there are two types as follows. For $x = y = v - 1$ the natural cut-block $A_{v-1,v-1} = \bar{K}_2 + K_{v-1}$ has diameter 2, is of order $v+1$, and is clearly minimum. A $(d, 1, v)$-graph G can have a natural cut-block $A_{v-1,v-1}$ only if G has bridges and hence only if v is odd. The other natural cut-block $A_{k,v-k}$ $(2 \leq k \leq v-2)$ is of diameter 3 and order $v+2$ and is formed by adjoining a point p_0 to each of the k points of the subgraph K_k and adjoining a point p_3 to each of the $v-k$ points of K_{v-k} in the join $K_k + K_{v-k}$ of order v of the complete graphs K_k and K_{v-k}. $A_{k,v-k}$ is minimum since the number of points adjacent to p_0, p_3 is equal to the degree k of p_0 and the degree $v-k$ of p_3, respectively, and $A_{k,v-k} - p_0 - p_3$ is of order v. Note that by Lemma 1, a $(d, 1, v)$-graph G with v even can have no cut-block $A_{k,v-k}$ with k odd. In any event, we have:

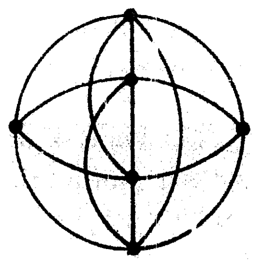
Theorem 2. *The natural cut-blocks are* $A_{v-1,v-1}$ *of diameter* 2 *and order* $v+1$, *where* v *is odd, and* $A_{k,v-k}$ $(2 \leq k \leq v-2)$ *of diameter* 3 *and order* $v+2$, *for both odd and even* v.

The natural cut-blocks $A_{4,4}$ and $A_{2,3}$ $(= A_{3,2})$ for $v = 5$ are shown in Figs. 9(a) and (b), and the natural cut-block $A_{2,2}$, which is Klee and Quaife's *diamond* [27], in Fig. 9(c).

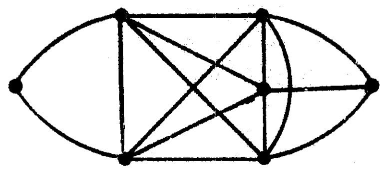
We call a string of natural cut-blocks $A_{v-k,k}$ $(2 \leq k \leq v-2)$, or a string of natural cut-blocks $A_{v-1,v-1}$ alternating with bridges, a *natural string*. The *length* of a string is its diameter, and its two *ends* are its two points of degree less than v. Since the natural cut-block construction guarantees that the number of points equidistant from either end of a natural string is minimum, such a string of any given length has the minimum order for a string of that length.

4. Minimum $(d, 1, v)$-graphs

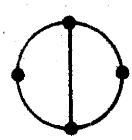
Suppose that G is a minimum $(d, 1, v)$-graph. By Lemma 2, there is no end-block of diameter 1. And since every such graph G of degree 3 must have at



(a) $A_{4,4}$ for $v = 5$



(b) $A_{2,3}$ $(= A_{3,2})$ for $v = 5$



(c) $A_{2,2}$ for $v = 3$

Fig. 9. Natural cut-blocks.

least one bridge in that it has a cutpoint, we have:

Lemma 7. *A minimum $(d, 1, v)$-graph of degree $v \geqslant 4$ has $d > 3$; and one of degree $v = 3$ has $d > 4$.*

For example, the complete graph K_{v+1} is the unique minimum $(1, c, v)$-graph, and has $c = v$. And for $v = 3$ there are a total of seven minimum $(d, c, 3)$-graphs for $d < 4$, all of which are nonseparable [27].

We shall next show that every minimum $(d, 1, v)$-graph G is a string, and thus has exactly two end-blocks, each of its cut-blocks thus a cut-block A_{x_1, x_2}.

Minimality

We show that a $(d, 1, v)$-graph G that is not a string is not minimum, as follows.

If G is not a string, it has at least one cutpoint p_0 which separates the graph into three or more components $G_1, G_2, \ldots, G_m$ $(m \geqslant 2)$ of diameters $d_1, d_2, \ldots, d_m$, respectively, so that G has at least m end-blocks. Since G has diameter d we may assume that $d_1 + d_2 = d$ and that $d_i + d_j \leqslant d$ for $2 \leqslant i < j \leqslant m$. Suppose that G is minimum, and hence that each component G_i $(3 \leqslant i \leqslant m)$ is minimum since otherwise G_i could be replaced by a component G_i' of lower order and, if needs be, of lower diameter, contradicting the hypothesis that G is minimum. It follows immediately that each component G_i $(3 \leq i \leq m)$ is either a minimum end-block B_{v-k} $(2 \leq k \leq v-2)$ of diameter 2, and hence of order $v+2$ for v even and $v+3$ for v odd, or has v odd and is the string of a bridge and a minimum end-block B_{v-1} of diameter 2, and hence of order $v+3$. Let $G_m' = G_m - p_0$ and $G_0 = G - G_m'$, so that p_0 is of degree less than v in G_0. We may assume without any loss in generality that p_0 has degree at least 2 in G_0, since if it is the end-point of a bridge, that bridge may be assumed to be in G_m. If G_m is the string of a bridge and a minimum end-block B_{v-1}, so that G_m' is of order $v+2$, then p_0 is of degree $v-1$ in G_0. Suppose that G is of order n and hence that G_0 is of order $n-(v+2)$. Let $G_0' = G_0 - p_0$, so that each point in the set E_{v-1} of the $v-1$ points of G_0 that are adjacent to p_0 is of degree $v-1$ in G_0'. Whether or not p_0 is in a longest geodesic which determines the diameter d of G, the following technique may be applied for obtaining a $(d, 1, v)$-graph G' of lower order than G when $v \geq 7$, thus dismissing the hypothesis that G is minimum. Further, G' has one less cutpoint than G, so that successive application of the technique results in a minimum graph, each cutpoint of which separates the graph into exactly two components, the graph thus a string.

Suppose p_0 is in a longest geodesic of G and that p_0', p_0'' are the two points of E_{v-1}, each adjacent to p_0 in G, that are in that geodesic (there being exactly two such points). (If p_0 is not in a longest geodesic, simply let p_0', p_0'' be any two points of E_{v-1}). Let $E_{v-3} = E_{v-1} - p_0' - p_0''$, which is of even order since v is odd. Adjoin p_0' and p_0'' to a point of a graph G_0'' of order $\frac{1}{2}(v-1)$ (≥ 6) and degree $\frac{1}{2}(v-7)$ (≥ 0). G_0'' exists since it is clearly a subgraph of the complete graph K_x of order $x = \frac{1}{2}(v-1)$; and adjoin each of the $\frac{1}{2}(v-3)$ pairs of points of E_{v-3} to different other points of G_0'' as illustrated in Fig. 10. Then join each point of G_0'' to each of the points of the complete graph G_0''' of order $\frac{1}{2}(v+3)$ and degree $\frac{1}{2}(v+1)$, as in Fig. 10, to form a graph G'. The graph G' is a $(d, 1, v)$-graph with one less

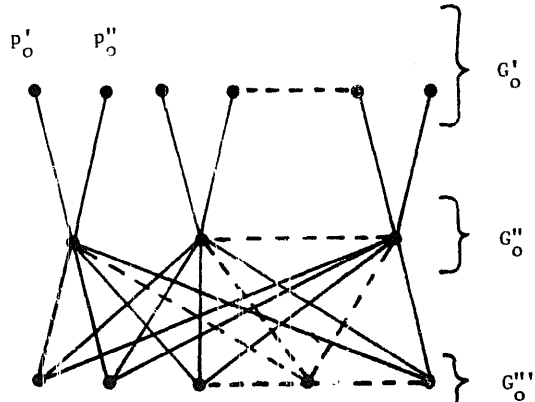


Fig. 10. Minimalization of a $(d, 1, v)$-graph ($v = 7$ and odd).

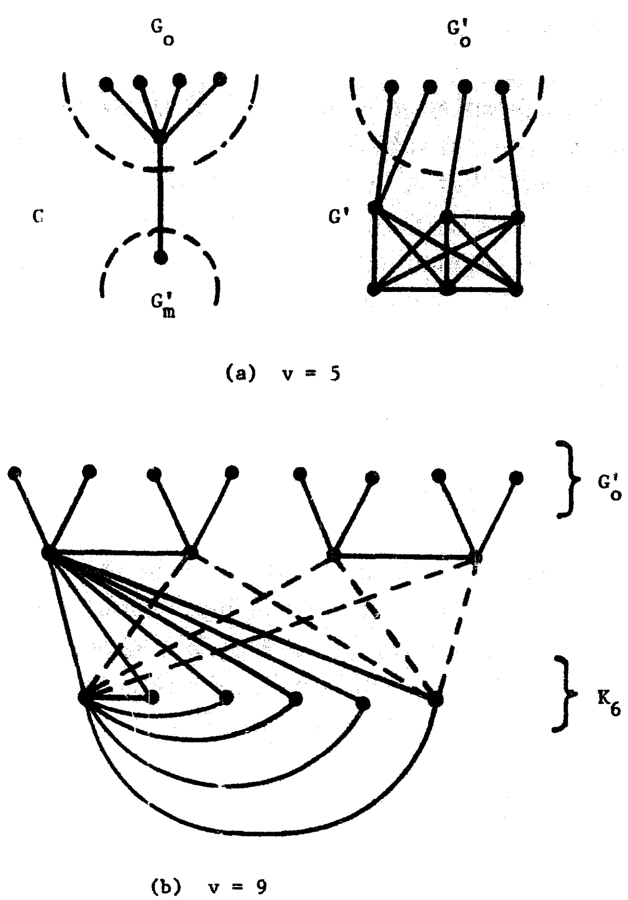


Fig. 11. Examples of minimalization (v odd).

cutpoint than G and is of order equal to the order $n-(v+3)$ of G_0' plus the sum $\frac{1}{2}(v-1)+\frac{1}{2}(v+3)$ of the orders of G_0'' and G_0''', i.e. $n-2$, or two less than the order of G. This dismisses the assumption that G is minimum.

We may repeat the process to successively eliminate each of the components G_i $(3 \leq i \leq m-1)$, the end result being that G is minimal if and only if it has exactly two end-blocks and is therefore a string. The case when $v=3$ has been treated separately by Myers [40], and the elimination technique for $v=5$ is shown pictorially in Fig. 11(a). The technique is illustrated in Fig. 11(b) for the case when $v=9$.

For the case when G_m is an end-block, we may again assume without any loss in generality that p_0 is in a longest geodesic in G. When v is even, so also is the degree of p_0 in both G_m and G_0. When v is odd, p_0 is of even (odd) degree in G_m when it has odd (even) degree in G_0.

We first construct an *augmentation subgraph* A, as follows. Suppose p_0 has degree k in G_0. Join a point p to each of the $v-k$ points of $\bar{K}_{v-k}$ in the complete bipartite graph $\bar{K}_{v-k}+\bar{K}_q$, where q is to be determined, thus forming a subgraph A' of A. Each of the points of $\bar{K}_{v-k}$ and $\bar{K}_q$ are to have degree $v-2$ in A, so that each of the points of $\bar{K}_{v-k}$ are to be augmented in degree by an amount $(v-2)-(q+1)=v-3-q$ and those of $\bar{K}_q$ by $(v-2)-(v-k)=k-2$, i.e. each point of $\bar{K}_q$ must be adjacent to $k-2$ points of $\bar{K}_q$ in addition to the $v-k$ points of $\bar{K}_{v-k}$. We shall show that it suffices to let $q=k-1$, which is ≥ 2. Assume first that both k and v are odd. Let A be the graph formed from A' by adjoining each pair of points of $\bar{K}_q$ (so that the subgraph which spans the points of $\bar{K}_q$ in A is the complete graph K_q) and augmenting the points of $\bar{K}_{v-k}$ in degree by an amount $(v-2)-(q+1)=v-(k+2)$, which is ≥ 0 since $v-k\geq 2$ when G_m is a minimum end-block. Since k and v are odd, $v-(k+2)$ is even; and since $v-k$ is also even, the subgraph which spans the points of $\bar{K}_{v-k}$ in A is to be the line-disjoint union of $\frac{1}{2}(v-k-2)$ cycles each of order $v-k$, which is always realizable. We next superimpose A on G_0, to form a graph G', as follows. We may assume that the point p_0 is in a longest geodesic in G (since the augmentation of G_0 by A produces the same end result when p_0 is not in such a longest geodesic). Let p_0', p_0'' be the two points adjacent to p_0 in G_0 that are in that longest geodesic. Let $\bar{K}_q=\bar{K}_{k-1}$ be the mid-points of $k-1$ of the k lines incident with p_0, two of which are to be the lines (p_0, p_0') and (p_0, p_0''). Let $\bar{K}_{v-k}$ be the mid-points of $v-k$ of the lines incident with the k points of the set E_k of points that are adjacent to p_0 in G_0 but which are not incident with p_0, two of these $v-k$ lines to be those in the longest geodesic in G. And let p_0 be the point p of A. Adjoin these several points by lines as in A, thus superimposing A on G_0, to form a graph G' in which each

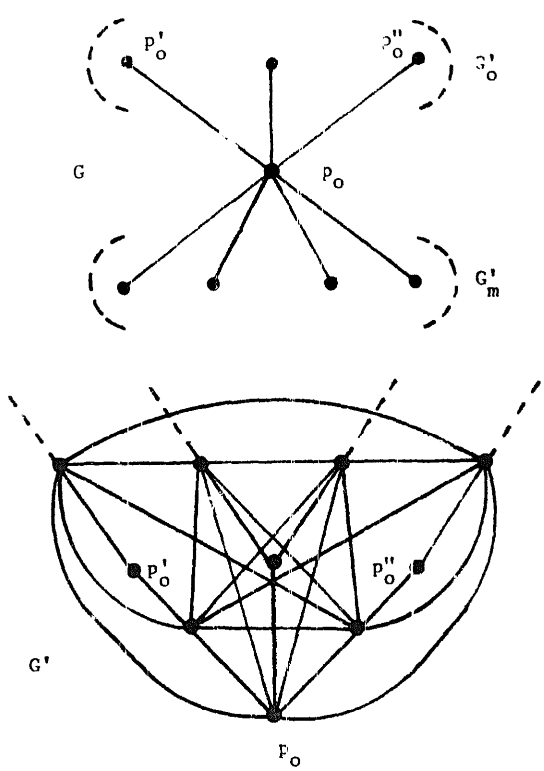
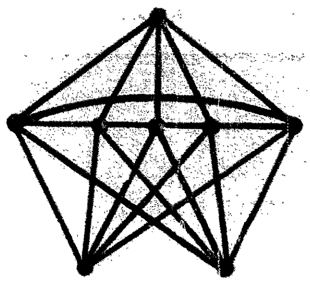
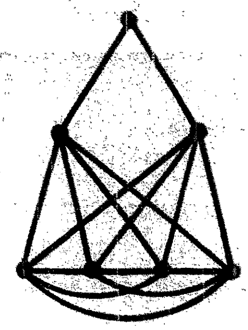


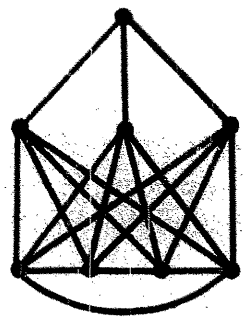
Fig. 12. Minimization by augmentation.



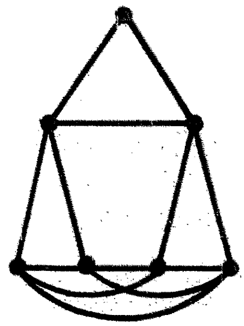
(a) v = 7, k = 2



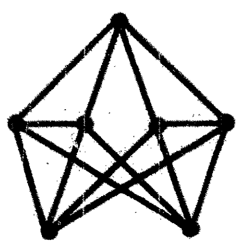
(b) v = 7, k = 5



(c) v = 7, k = 4



(d) v = 6, k = 4



(e) v = 6, k = 2

Fig. 13. Augmentation graphs.

of the points of A has degree v. Since G_m is of order $v+3$ for v odd, and since A is of order $v+1$ and the point p of A is the point p_0 of G_0, G' is of order two less than the order of G and is a $(d, 1, v)$-graph with one less cutpoint than G. The construction is such that the diameter of G' is the diameter d of G. Hence, G is not minimum.

The augmentation process is illustrated in Fig. 12 for $v=7$ with $k=3$.

We do likewise to replace each of the components G_i ($3 \leq i \leq m-1$) of G until finally, the resulting $(d, 1, v)$-graph has exactly two components for each of its cutpoints, has exactly two end-blocks and is thus a string, and is of order $m-2$ less than the order n of G.

The procedure is exactly the same for v odd and k even and for v even (in which case k cannot be odd), being a routine matter of evaluating q for each case. Suffice it, therefore, to simply give some examples. The augmentation graphs A are shown in Figs. 13(a), (b), (c), (d), (e) for $(v, k)=(7, 2)$, $(7, 5)$, $(7, 4)$, $(6, 4)$, (6, 2), respectively. Note that A is of order $v+1$ when k is odd, and v when k is even, irrespective of v. Thus, G' is of order one less than G when k is even, and two less otherwise.

In summary, we have:

Theorem 3. *Every minimum* $(d, 1, v)$*-graph is a string.*

Minimum $(d, 1, v)$*-graphs for* $4<d<G$.

As noted earlier, a minimum $(4, 1, v)$-graph G is formed by stringing an end-block B_{v-2l} of diameter 2 with an end-block B_{2k} also of diameter 2 $(2\leq 2k\leq v-2)$, and is of order $2v+3$, $2v+4$ according as v is even, odd, by Table 2. For $v=4$, there is one realization. For v even and greater than 4 there are thus at least as many realizations as there are partitions of v into two parts $2k$, $v-2k$, namely at least $\frac{1}{4}v$ realizations for $v=4j$ $(j>1)$ and $\frac{1}{4}(v+2)$ realizations for $v=(2j+1)2$ $(j\geq 1)$. When v is odd, so that $v=2j+1$ $(j\geq 1)$, there is one realization for $v=5$ and at least $\frac{1}{2}(v-3)$ realizations for $v>5$.

Lemma 8. *Every minimum* $(4, 1, v)$*-graph* G $(v\geq 4)$ *is of order* $2v+3$, $2v+4$ *for* v *even, odd, respectively.*

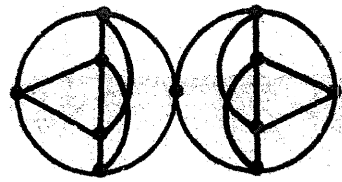
The unique minimum $(4, 1, v)$-graphs for $v=4$ and $v=5$ are shown in Figs. 14(a) and 14(c) and a minimum $(4, 1, 6)$-graph in Fig. 14(b).

A $(5, 1, v)$-graph G is either a string of two end-blocks B_{v-1}, each of diameter 2, and a bridge, or a string of two end-blocks B_{v-2}, B_2 $(2\leq 2k\leq v-2)$, one of diameter 2 and the other of diameter 3. From Table 2 we see that:

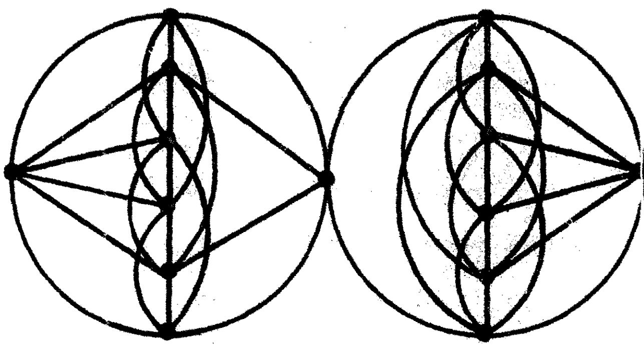
Lemma 9. *The string of order* $2v+4$ *of two minimum end-blocks* B_{v-1} *each of order 2, and a bridge, is the unique minimum* $(5, 1, v)$*-graph for* $v\geq 3$ *and odd. The string of order* $2v+5$ *of a minimum end-block* B_2 *of diameter 3 and a minimum end-block* B_{v-2} *of diameter 2 is the unique minimum* $(5, 1, v)$*-graph for* $v\geq 4$ *and even.*

In similar fashion:

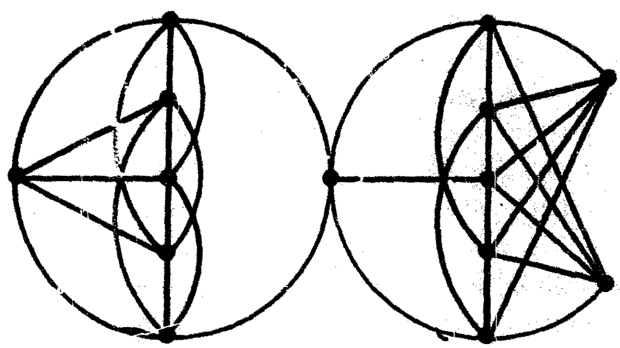
Lemma 10. *The string of a bridge and two minimum end-blocks* B_{v-1}, *one of diameter 2 and the other of diameter 3, each string of a minimum end-block* B_{2k} $(2\leq 2k\leq v-3)$ *of diameter 2 and a minimum end-block* B_{v-2k} *of diameter 4, and each string of a minimum end-block* B_{2k} $(2\leq 2k\leq v-3)$ *and a minimum end-block* B_{v-2k}, *each of diameter 3, is a minimum* $(6, 1, v)$*-graph for odd* $v=2j+1$ $(j\geq 1)$, *and is of order* $3v+3$. *There is one such graph for* $v=3$ *and there are* $(v-2)$ *such graphs for* $v=2j+1$ $(j\geq 2)$. *Each string of a minimum end-block* B_{2k} $(2\leq 2k\leq v-2)$ *and a minimum end-block* B_{v-2k}, *each of diameter 3, is a*



(a) The unique minimum (4,1,4)-graph



(b) Minimum (4,1,6)-graph



(c) The unique minimum (4,1,5)-graph

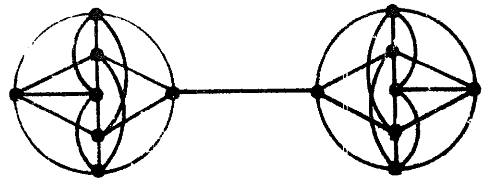
Fig. 14. Minimum $(d, 1, v)$-graphs of diameter 4.

minimum $(6, 1, v)$-graph for v even. There are $v/4$ such graphs for $v = 4j$ ($j \geq 1$) and $\frac{1}{4}(v+2)$ for $v = (2j+1)2$ ($j \geq 1$), each of order $3v+3$.

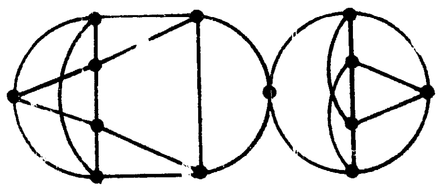
The unique minimum $(5, 1, v)$-graphs for $v = 4$ and $v = 5$ are shown in Figs. 15(a), (b). The minimum $(6, 1, 4)$-graph which is the string of two end-blocks B_2 is also unique.

Minimum $(d, 1, v)$-graphs ($d > 4$)

By Lemma 6, every minimum end-block B_{x_1} of diameter 5 has a cut-point and is of order $2v+3$ when x_1 is even and $2v+4$ when x_1 is odd. As illustrated for



(a) The unique minimum (5,1,5)-graph



(b) The unique minimum (5,1,4)-graph

Fig. 15. Minimum (5, 1, v)-graphs.

$x = 3$ and $v = 5$ in Fig. 7, B_{x_1} is a string of a natural cut-block of diameter 3 and order $v+2$ with an end-block B_{x_1} of diameter 2 and order $v+2$ when x_1 is even and $v+3$ when x is odd (and therefore v also odd). A string of k (≥ 0) natural cut-blocks A_{x_1,x_2} each of length 3 and order $v+2$ has length $3k$ and order $k(v+2)-(k-1) = k(v+1)+1$, and is clearly minimum for a string of this length. It follows immediately that the $(3k+4, 1, v)$-graph obtained on terminating such a string with two end-blocks each of diameter 2 is minimum, since it may alternately be regarded as the minimum graph obtained on terminating a minimum string of length $(3k-2)$ of $k-2$ natural cut-blocks A_{x_1,x_2} with two minimum end-blocks each of diameter 5 when $k \geq 2$, or as a string of two end-blocks one of diameter 5 and the other of diameter 2 when $k = 1$. Since every minimum end-block B_x is of order $v+2$ for x even and $v+3$ for x odd, then every minimum $(3k+4, 1, v)$-graph ($k \geq 0$) is of order $(k+2)(v+1)+1$ for v even and $(k+2)(v+1)+2$ for v odd.

Alternately, a string of length $3j+1$ of natural cut-blocks $A_{v-1,v-1}$ (v odd) alternating with $j+1$ bridges is minimum for a string of that length, its order being $j(v+1)+2$. Similarly to the above case, we see that whenever $k \geq 1$, such a string terminated in two end-blocks B_{v-1}, one of diameter 2 and the other of diameter 4, is also a minimum $(3k+4, 1, v)$-graph when v is odd.

In similar fashion, for v odd a minimum string of $k+1$ bridges and k natural cut-blocks $A_{v-1,v-1}$ ($k \geq 0$) terminated in two minimum end-blocks B_{v-1} each of diameter 2 is a minimum $(3k+5, 1, v)$-graph, its order being $(2v+4)+k(v+1) = (k+2)(v+1)+2$. For v even, a string of k (≥ 0) natural cut-blocks A_{x_1,x_2} terminated in two minimum end-blocks B_{x_1}, B_{x_2}, one of diameter 2 and one of diameter 3, is a minimum $(3k+5, 1, v)$-graph, its order being $(2v+6)+k(v+2)-(k+1) =$

Table 3. Order n of a minimum $(d, 1, v)$-graph with all natural cut-blocks $A_{v-1,v-1}$ and (v odd); $(d) = (a+b)$; $n = n_a + n_b$.

a	b / b_1, b_2 / n_b / n_a	4 / 2, 2 / $2v+4$	5 / 2, 3 / $3v+3$	6 / 2, 4 / $3v+5$
1	0	$2v+4$ (5)	$3v+3$ (6)	$3v+5$ (7)
4	$v+1$	$3v+5$ (8)	$4v+4$ (9)	$4v+6$ (10)
7	$2v+2$	$4v+6$ (11)	$5v+5$ (12)	$5v+7$ (13)

b_1, b_2 are the end-block diameters; $b = b_1 + b_2$.
a is the sum of the cut-block and bridge diameters.
n_b is the sum of the orders of the end-blocks.
n_a is the sum of the orders of the cut-blocks.

$(k+2)(v+1)+3$. Further, for v odd a string of k (≥ 0) natural cut-blocks $A_{v-1,v-1}$ terminated in two minimum end-blocks B_{v-1}, one of diameter 2 and the other of diameter 3, is a minimum $(3k+6, 1, v)$-graph, its order being $(3v+2)+k(v+1) = (3+k)(v+1)$. A minimum $(3k+6, 1, v)$-graph with v even is obtained by stringing two minimum end-blocks B_{x_1}, B_{x_2} each of diameter 3 with a string of k (≥ 0) natural cut-blocks A_{x_1,x_2}, the graph then being again of order $(3v+4)+k(v+2)-(k+1) = (3+k)(v+1)$. An alternate realization of a $(3k+6, 1, v)$-graph G with v odd is to string two minimum end-blocks B_{x_1}, B_{x_2}, one of diameter 2 and the other

Table 4. Order n of a minimum $(d, 1, v)$-graph with all natural cut-blocks A_{x_1,x_2} and v odd; $(d) = (a+b)$; $n = n_a + n_b + n_c$

a	n_a	b / b_1, b_2 / n_b / n_c	4 / 2, 2 / $2v+5$	6 / 2, 4 / $3v+5$
0	0	-1	$2v+4$ (4)	$3v+3$ (6)
3	$v+2$	-2	$3v+5$ (7)	$4v+4$ (9)
6	$2v+4$	-3	$4v+6$ (10)	$5v+5$ (12)

b_1, b_2 are the end-block diameters; $b = b_1 + b_2$.
a is the sum of the cut-block diameters.
n_b is the sum of the orders of the end-blocks.
n_a is the sum of the orders of the cut-blocks.
n_c is the number of cutpoints, times -1.

of diameter 4 with k natural cut-blocks A_{x_1,x_2}, the order of G thus being $(3v+4)+(v+2)k-(k+1)=(3+k)(v-1)$, as before. It is interesting to observe that there is no minimum $(3k+5, 1, v)$-graph all of whose cut-blocks are natural cut-blocks A_{x_1,x_2}. For $d=3k+4$ and $d=3k+6$, however, this is not the case. The values of $3k+4$, $3k+5$ and $3k+6$ for $k=0,1,2,3,\ldots$ span all integers $4,5,6,7,\ldots$, and hence there are no other cases to consider.

These findings summarize as:

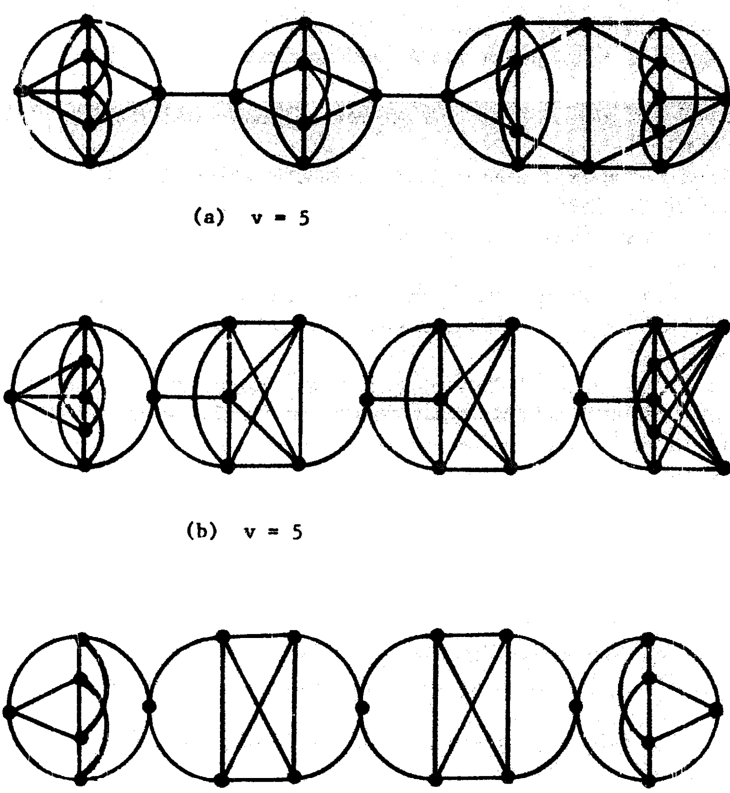
Theorem 4. *Every minimum $(3k+j, 1, v)$-graph $(k \geq 0,\ 4 \leq j \leq 6)$ is of order $(k+2)(v+1)+1$ for v even and $(k+2)(v+1)+2$ for v odd when $j=4$, of order $(k+2)(v+1)+3$ for v even and $(k+2)(v+1)+2$ for v odd when $j=5$, and of order $(k+3)(v+1)$ when $j=6$.*

We display the results of Theorem 4 in tabular form in Table 3 for $3k+j=d=5,6,7,\ldots,13$ for v odd for a minimum $(d, 1, v)$-graph G all of whose cut-blocks are natural cut-blocks $A_{v-1,v-1}$. The diameter-pairs b_1, b_2 of the two minimum end-blocks B_{v-1} in G, and the sums $b=b_1+b_2$, are given as column indices in the table, and the lengths a of the string of cut-blocks and bridges are given as row indices, so that $d=a+b_1+b_2$. The sum of the orders of the two end-blocks B_{v-1} is given below the b_1, b_2 heading in each column, and the total order $k(v+1)$ $(k \geq 0)$ of the k cut-blocks $A_{v-1,v-1}$ in G is entered next to a in each row. Each entry in the table matrix gives the order of G as a function of v, and also gives the diameter of G in parentheses. For example, we see from the entry in the first column and the second row of the matrix that the order of G with a diameter 8 is $3v+5$. Similarly, G is of order $6v+6$ when $d=15$.

Table 5. Order n of a minimum $(d, 1, v)$-graph with all natural cut-blocks A_{x_1,x_2} and v even; $(d)=(a+b)$; $n=n_a+n_b+n_c$.

		b	4	5	6
		b_1, b_2	2, 2	2, 3	3, 3
		n_b	$2v+4$	$2v+6$	$3v+4$
a	n_a	n_c			
0	0	-1	$2v+3$ (4)	$2v+5$ (5)	$3v+3$ (6)
3	$v+2$	-2	$3v+4$ (7)	$3v+6$ (8)	$4v+$[illegible] (9)
6	$2v+4$	-3	$4v+5$ (10)	$4v+7$ (11)	$5v+5$ (12)

b_1, b_2 are the end-block diameters; $b=b_1+b_2$.
a is the sum of the cut-block diameters.
n_b is the sum of the orders of the end-blocks.
n_a is the sum of the orders of the cut-blocks.
n_c is the number of cut-points, times -1.



(a) v = 5

(b) v = 5

(c) v = 4

Fig. 16. Minimum (10, 1, v)-graphs.

Table 4 lists the corresponding information for v odd when each cut-block in G is a natural cut-block A_{x_1,x_2}, and Table 5 likewise for graphs which have v even.

Two different minimum $(10, 1, 5)$-graph configurations are shown in Figs. 16(a), (b), and a minimum $(10, 1, 4)$-graph in Fig. 16(c).

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